

\mathcal{CP} violation with an unbroken \mathcal{CP} transformation

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ABSTRACT: A \mathcal{CP} conserving $SU(3)$ gauge theory is spontaneously broken to T_7 by the vacuum expectation value (VEV) of a $\mathbf{15}$ -plet. Even though the $SU(3)$ – \mathcal{CP} transformation is not broken by the VEV, the theory exhibits physical \mathcal{CP} violation in the broken phase. This is because the $SU(3)$ – \mathcal{CP} transformation corresponds to the unique order-two outer automorphism of T_7 , which is not a physical \mathcal{CP} transformation for the T_7 states, and there is no other possible \mathcal{CP} transformation. We explicitly demonstrate that \mathcal{CP} is violated by calculating a \mathcal{CP} odd decay asymmetry in the broken phase. This scenario provides us with a natural protection for topological vacuum terms, ensuring that $\theta G_{\mu\nu} \tilde{G}^{\mu\nu}$ is absent even though \mathcal{CP} is violated for the physical states of the model.

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1 Introduction

Violation of the combined transformation of charge conjugation and parity (\mathcal{CP}) has been observed in decays and oscillations of K and B mesons [1, 2]. Complementary to that, violation of the time reversal transformation (\mathcal{T}) is implied by the \mathcal{CPT} theorem and has been verified independently [3]. Likewise, recent global fits of neutrino oscillation data point towards \mathcal{CP} violation also in the lepton sector [4]. Given that the measurement of leptonic mixing angles has entered its precision phase, and anticipating the near future measurement of neutrino masses, the experimental efforts to pin down all parameters which constitute the SM flavor puzzle is close to being completed.

Even though the observed fermion masses, mixings, and \mathcal{CP} violation can be consistently parametrized in the framework of 3×3 (CKM [5] or PMNS [6]) unitary mixing, the overall theoretical situation is in many ways unsatisfactory. First of all, \mathcal{CP} violation beyond the standard model is necessary in order to explain the baryon asymmetry of the universe [7]. Furthermore, \mathcal{CP} violation is observed only in flavor changing transitions mediated by the weak interaction, not in strong interactions, thereby giving rise to a severe fine-tuning problem of the $\bar{\theta}_{\text{QCD}}$ parameter. Ultimately, the sheer multitude of parameters in the flavor sector and their distinct pattern cries out for an underlying fundamental explanation. Therefore, more theoretical work is necessary in order to unveil the underlying structure of the flavor puzzle, including the origin of \mathcal{CP} violation.

From a formal point of view, \mathcal{CP} transformations are quite particular. While \mathcal{P} or \mathcal{C} transformations individually relate otherwise disconnected representations of the space-time and gauge symmetries of a model, a \mathcal{CP} transformation maps each irreducible representation (irrep) to *its own* complex conjugate representation. This implies that in theories with a real valued action, \mathcal{CP} cannot be broken *maximally* by simply leaving out or adding fields, in vast contrast to, for example, the parity transformation in a chiral theory. From a group-theoretical point of view this is reflected by the fact that \mathcal{CP} transformations are special outer¹ automorphisms of the continuous [8], discrete [9, 10] and space-time [11] symmetries of a model, which map the irreps of each group to their own complex conjugate representations [12]. This should be contrasted to other outer automorphisms, such as parity or charge conjugation for example, which map irreps to other irreps that may not be present in a model to begin with.

This notion defines \mathcal{CP} transformations as special automorphisms of symmetry groups. However, it is not guaranteed that such an automorphism exists for a given symmetry group [10]. While it is known that \mathcal{CP} outer automorphisms exist for simple Lie groups [8] and the Poincaré group [11] it has been pointed out that there are certain discrete groups which violate \mathcal{CP} by the intrinsic complexity of their Clebsch–Gordan coefficients [13]. In more detail, these so-called “type I” groups prohibit simultaneous $\mathbf{r}_i \leftrightarrow \mathbf{r}_i^*$ (for all i , labeling the irreps of the corresponding group) transformations [10]. That is, these type I groups do not allow for outer automorphisms which could be identified with physical \mathcal{CP} transformations for all of their irreps. If a model features such a group as global symmetry and a sufficiently

¹The automorphism must be outer if the corresponding symmetry group has complex representations.

large number of irreps, then \mathcal{CP} transformations are not possible in consistency with the symmetry group of the model. Explicitly it has been found that the necessarily complex Clebsch–Gordan coefficients of type I groups then enter \mathcal{CP} odd basis invariants, thereby giving rise to particle–antiparticle asymmetries in oscillations and decays [10]. Due to the fact that the arising physical \mathcal{CP} –odd complex phases are discrete and calculable, this phenomenon has been termed explicit geometrical \mathcal{CP} violation [13, 14].

In this context, an important fact is that type I groups can arise as subgroups of simple Lie groups. This gives rise to the puzzling situation in which a Lie group G allows for a perfectly well–defined \mathcal{CP} transformation, i.e. a particular complex conjugation outer automorphism, whereas the type I subgroup, $H \subset G$, does not. Thus, \mathcal{CP} is not conserved at the level of H , and the question arises when and how \mathcal{CP} is broken in a possible breaking of $G \rightarrow H$. Naïvely, one might expect that in a dynamical setting it should be the vacuum expectation value (VEV) which gives rise to \mathcal{CP} violation. Rather surprisingly, it turns out that this is not necessarily the case. We will show that the VEV which spontaneously breaks $G \rightarrow H$ does *not* break the complex conjugation outer automorphism, i.e. the VEV is \mathcal{CP} conserving. Nevertheless, physical \mathcal{CP} is violated at the level of H , and we will substantiate this claim by an explicit calculation of non–vanishing \mathcal{CP} odd basis invariants that give rise to a physical decay asymmetry. The way the conundrum is resolved is the following: The conserved outer automorphism which gives rise to \mathcal{CP} conservation at the level of G is conserved by the VEV and, hence, also a conserved outer automorphism at the level of H . Nevertheless, at the level of H this outer automorphism is no complex conjugation automorphism. Thus, once the physical states of a theory are H states, the conserved outer automorphism can no longer be interpreted as a physical \mathcal{CP} transformation. An anticipated distinct outer automorphism transformation, which *would* correspond to a physical \mathcal{CP} transformation at the level of H , is prohibited by the group structure of H , and it would also not be a consistent automorphism of G to begin with.

The investigation in this paper is based on an economic toy example. We use an $SU(3)$ gauge theory which is broken to the type I group T_7 by a the VEV of a complex scalar ϕ , transforming in the **15**–plet representation of $SU(3)$ with the Dynkin indices $(2, 1)$. We assume physical \mathcal{CP} conservation at the level of $SU(3)$ and show that the VEV does not break the corresponding outer automorphism. Yet we will find \mathcal{CP} violating decays of a physical scalar to gauge bosons in the broken phase.

The rest of the paper is organized as follows. In Section 2, we define our model. Section 3 details the spontaneous breaking of $SU(3)$ to T_7 . In Section 4, we discuss how the T_7 states are related to the states of the $SU(3)$ theory. In Section 5, we show explicitly that \mathcal{CP} is broken in the T_7 phase. Section 6 contains a comment on the θ term of the $SU(3)$ theory. Finally, Section 7 contains our conclusions. Some details are deferred to the appendices.

Name	SU(3) $\xrightarrow{\langle\phi\rangle}$	Name	T ₇
A_μ	8	Z_μ	1₁
		W_μ	3
ϕ	15	Re σ_0 , Im σ_0	1₀
		σ_1	1₁
		τ_1	3
		τ_2	3
		τ_3	3

Table 1. Physical fields before and after the symmetry breaking. The number of real degrees of freedom before and after SSB coincide as $16 + 30 = 24 + 22$.

2 The model

We consider an SU(3) gauge theory with the Lagrangean

$$\mathcal{L} = (D_\mu \phi)^\dagger (D^\mu \phi) - \frac{1}{4} G_{\mu\nu}^a G^{\mu\nu,a} - \mathcal{V}(\phi) , \quad (2.1)$$

with $D_\mu = \partial_\mu - ig A_\mu$ being the ordinary gauge covariant derivative and $G_{\mu\nu}^a$ the field strength tensors. The complex scalar ϕ is charged under SU(3), transforming in the **15**-plet representation. The scalar potential is given by

$$\mathcal{V}(\phi) = -\mu^2 \phi^\dagger \phi + \sum_{i=1}^5 \lambda_i \mathcal{I}_i^{(4)}(\phi) , \quad (2.2)$$

where we take $\mu^2 > 0$, and the $\mathcal{I}_i^{(4)}$ denote the five independent completely symmetric quartic SU(3) invariants in the contraction $\overline{\mathbf{15}} \otimes \mathbf{15} \otimes \overline{\mathbf{15}} \otimes \mathbf{15}$. All other invariant vanish. These invariants are real, which allows us to take $\lambda_i \in \mathbb{R}$ without loss of generality. Further details on the derivation of the invariants can be found in Appendix A. There is a region in the parameter space in which this potential has a global minimum which gives rise to a VEV $\langle\phi\rangle$ that spontaneously breaks $\text{SU}(3) \rightarrow \text{T}_7$ [15, 16]. The linear symmetry of the vacuum is T₇, a discrete group with 21 elements. The physical spectrum after spontaneous symmetry breaking (SSB) is given in Table 1. We will give further details of T₇, including its embedding in SU(3) and branching rules in Section 3.1.

The action derived from (2.1) is automatically invariant under the simultaneous outer automorphisms of SU(3) and the Lorentz group under which the gauge and scalar fields transform as

$$A_\mu^a(x) \mapsto R^{ab} \mathcal{P}_\mu^\nu A_\nu^b(\mathcal{P}x) , \quad (2.3a)$$

$$\phi_i(x) \mapsto U_{ij} \phi_j^*(\mathcal{P}x) . \quad (2.3b)$$

Here $\mathcal{P} = \text{diag}(1, -1, -1, -1)$ is the usual spatial reflection, while R and U are the representation matrices of the outer automorphism of SU(3) fulfilling

$$R_{aa'} R_{bb'} R_{cc'} f_{a'b'c'} = f_{abc} , \quad \text{and} \quad U(-\mathbf{t}_a^T) U^{-1} = R_{ab} \mathbf{t}_b . \quad (2.4)$$

T_7	C_{1a}	C_{3a}	C_{3b}	C_{7a}	C_{7b}
	e	b	b ²	a	a ³
$\mathbf{1}_0$	1	1	1	1	1
$\mathbf{1}_1$	1	ω	ω^2	1	1
$\bar{\mathbf{1}}_1$	1	ω^2	ω	1	1
$\mathbf{3}$	3	0	0	η	η^*
$\bar{\mathbf{3}}$	3	0	0	η^*	η

Table 2. Character table of T_7 . The arrows illustrate the action of the unique \mathbb{Z}_2 outer automorphism of T_7 . We use the definitions $\omega := e^{2\pi i/3}$ and $\eta = \rho + \rho^2 + \rho^4$ with $\rho := e^{2\pi i/7}$.

Here, \mathbf{t}_a and f_{abc} , with the usual relation $[\mathbf{t}_a, \mathbf{t}_b] = i f_{abc} \mathbf{t}_c$, are the generators and structure constant of the Lie algebra, respectively. At the level of $SU(3)$, this outer automorphism maps all irreps to their own complex conjugate and, therefore, is the most general possible physical \mathcal{CP} transformation. The transformation is conserved by the action implying that the model is \mathcal{CP} symmetric in the unbroken phase. In addition, the VEV fulfills $U \langle \phi \rangle^* = \langle \phi \rangle$ and, therefore, does not break the outer automorphism. Surprisingly, we will find that even though the $SU(3)$ – \mathcal{CP} transformation is not broken, physically, \mathcal{CP} is no longer conserved once $SU(3)$ gets broken to T_7 .

As will be detailed below, the unique \mathbb{Z}_2 outer automorphism of $SU(3)$ corresponds to the unique \mathbb{Z}_2 outer automorphism of T_7 . Therefore, the actual symmetry breaking chain of the model is given by

$$SU(3) \rtimes \mathbb{Z}_2 \xrightarrow{\langle \phi \rangle} T_7 \rtimes \mathbb{Z}_2. \quad (2.5)$$

Crucially, the \mathbb{Z}_2 transformation on the left-hand side corresponds to a \mathcal{CP} transformation, while the identical transformation on the right-hand side cannot be interpreted as \mathcal{CP} . In more detail, at the level of T_7 , the unique \mathbb{Z}_2 outer automorphism acts as (cf. also Table 2)

$$\text{Out}(T_7) : \quad \mathbf{1}_1 \longleftrightarrow \mathbf{1}_1, \quad \bar{\mathbf{1}}_1 \longleftrightarrow \bar{\mathbf{1}}_1, \quad \mathbf{3} \longleftrightarrow \bar{\mathbf{3}}. \quad (2.6)$$

This transformation does *not* map all T_7 irreps to their own complex conjugate representation. Therefore, it does not correspond to a physical \mathcal{CP} transformation if triplet and non-trivial singlet representations of T_7 are present simultaneously. This is the case in the given model. That is, the model does not allow for a physical \mathcal{CP} transformation at the level of T_7 . Hence, the setting exhibits \mathcal{CP} violating processes once $SU(3)$ gets broken to T_7 . For definiteness, we will explicitly show in Section 5 that there is a \mathcal{CP} asymmetry in the decay of a heavy charged scalar to massive gauge bosons of the broken $SU(3)$ symmetry.

Here it should also be noted that any supposed physical \mathcal{CP} transformation $\mathbf{r}_{T_7} \leftrightarrow \mathbf{r}_{T_7}^*$ is inconsistent with the structure of T_7 . For this reason the group has been classified as a finite group of “type I” in [10]. In fact, a supposed transformation $\mathbf{r}_{T_7} \leftrightarrow \mathbf{r}_{T_7}^*$ is not a consistent automorphism at the level of $SU(3)$ to begin with. Imposing this transformation as a symmetry nonetheless, enforces $g = \lambda_i = 0$, i.e. forbids all interactions.

3 SU(3) and T₇ subgroup

3.1 Embedding

The discrete group T₇ can be generated by two elements with the presentation

$$\langle a, b \mid a^7 = b^3 = e, b^{-1} a b = a^4 \rangle . \quad (3.1)$$

For the triplet representation we choose a basis in which a and b are represented by

$$A = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho^4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} , \quad (3.2)$$

respectively, with $\rho := e^{2\pi i/7}$.

The group elements of SU(3) for an arbitrary representation \mathbf{r} are given by

$$X^{(\mathbf{r})} = \exp \left(i \alpha_a \mathbf{t}_a^{(\mathbf{r})} \right) , \quad (3.3)$$

where $\mathbf{t}_a^{(\mathbf{r})}$ denote the generators of the representation \mathbf{r} and α_a the parameters. We emphasize that there are two generally different spaces which are relevant here. While both, $[X^{(\mathbf{r})}]_{ij}$ and $[\mathbf{t}_a^{(\mathbf{r})}]_{ij}$, live in the “ ij –space” there is also the “ a ” or adjoint space in which the parameters $\vec{\alpha}$ as well as the gauge bosons live. This is important, because it is possible to choose bases independently for each of these spaces.²

For practical reasons, we do not work in the standard Gell–Mann basis of the adjoint space but follow the basis choice of Fonseca’s `Mathematica` package `SusyNo` [17]. The generators of the fundamental representation $\mathbf{r} = \mathbf{3}$ used in this work are specified in Appendix B.

The discrete subgroup T₇ is embedded into SU(3) via the irreducible triplet representation. In the given basis we find that A and B are obtained from (3.3) by the choice of parameters

$$\vec{\alpha}^{(A)} = \frac{2\pi}{7} (0, 0, 0, 0, 0, 0, \sqrt{3}, 5) \quad \text{and} \quad \vec{\alpha}^{(B)} = \frac{4\pi}{3\sqrt{3}} (0, 0, 1, 1, 1, 0, 0, 0) . \quad (3.4)$$

The branching rules of representations under SU(3) \rightarrow T₇ can be calculated with the help of [18]. Branchings relevant to this work are

$$\mathbf{8} \rightarrow \mathbf{1}_1 \oplus \bar{\mathbf{1}}_1 \oplus \mathbf{3} \oplus \bar{\mathbf{3}} , \quad (3.5a)$$

$$\mathbf{15} \rightarrow \mathbf{1}_0 \oplus \mathbf{1}_1 \oplus \bar{\mathbf{1}}_1 \oplus \mathbf{3} \oplus \mathbf{3} \oplus \bar{\mathbf{3}} \oplus \bar{\mathbf{3}} . \quad (3.5b)$$

²Only for special transformations — which are precisely the group transformations — it is possible to compensate transformations of the ij –space by transformations of the adjoint space.

3.2 Outer automorphism

Fonseca’s basis choice has the virtue of being a \mathcal{CP} basis [8], meaning that $U = \mathbb{1}$ in (2.4) irrespective of the specific representation. At the same time, the outer automorphism transformation in the adjoint space in this basis is given by

$$R = \text{diag}(-1, -1, 1, 1, 1, -1, -1, -1), \quad (3.6)$$

which can easily be computed from (2.4). We stress that in contrast to the U ’s, it is not possible to choose a basis in which R is trivial, for a non-trivial automorphism.

Since we ultimately will break $\text{SU}(3)$ to T_7 , it is convenient to rotate the ij -space of the scalar representation to a T_7 -diagonal basis, in which (3.5b) is explicitly realized for the T_7 generators parametrized by (3.4). The corresponding matrix for the basis change is given in Appendix B. Most importantly, the transformation matrix of the outer automorphism, U (cf. Equation (2.4)), is not invariant under such a basis change.³ In particular, we find that

$$U_{\mathbf{15}}^{(\text{T}_7)} = \mathbb{1}_1 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \mathbb{1}_{12}, \quad (3.7)$$

where $\mathbb{1}_n$ denotes the n -dimensional unit matrix. This shows that the $\text{SU}(3)$ outer automorphism acts like (2.6) on the T_7 representations.

Analogously, it is possible to choose a basis for the gauge bosons in the a -space such that (3.5a) is explicitly realized. The corresponding basis change is also given in Appendix B. We will see later that this basis also corresponds to the physical basis for the gauge bosons.

Using the basis changes in reverse, one can also show that the naïve \mathcal{CP} transformation at the level of T_7 (which would have to take $\mathbf{r}_i \rightarrow \mathbf{r}_i^* \forall \text{ T}_7 \text{ irreps } i$) does not fulfill (2.4) and, hence, does not correspond to a valid $\text{SU}(3)$ automorphism.

4 Physical states in the T_7 phase

In the T_7 -diagonal basis of $\text{SU}(3)$, and using unitary gauge, the scalar ϕ can be written as

$$\phi = \left(v + \phi_1, \frac{\phi_2}{\sqrt{2}}, \frac{\phi_2^*}{\sqrt{2}}, \phi_4, \phi_5, \phi_6, \frac{\phi_7}{\sqrt{2}}, \frac{\phi_8}{\sqrt{2}}, \frac{\phi_9}{\sqrt{2}}, \phi_{10}, \phi_{11}, \phi_{12}, \frac{\phi_7^*}{\sqrt{2}}, \frac{\phi_8^*}{\sqrt{2}}, \frac{\phi_9^*}{\sqrt{2}} \right), \quad (4.1)$$

featuring $22 = 30 - 8$ real degrees of freedom. In this basis, it is straightforward to identify the T_7 representations of the components as

$$\begin{aligned} \phi_1 &\hat{=} \mathbf{1}_0, & \phi_2 &\hat{=} \mathbf{1}_1, \\ T_1 &:= (\phi_4, \phi_5, \phi_6) \hat{=} \mathbf{3}, & T_2 &:= (\phi_7, \phi_8, \phi_9) \hat{=} \mathbf{3}, \\ \bar{T}_3 &:= (\phi_{10}, \phi_{11}, \phi_{12}) \hat{=} \bar{\mathbf{3}}. \end{aligned} \quad (4.2)$$

³Note that U under basis changes rotates with VUV^T rather than VUV^\dagger .

The VEV in this basis is simply given by

$$\langle \phi \rangle_1 = v \quad \text{and} \quad \langle \phi \rangle_i = 0 \quad \text{for} \quad i = 2, \dots, 15. \quad (4.3)$$

Minimizing the potential one finds that

$$|v| = \mu \times 3 \sqrt{\frac{7}{2}} \left(-7\sqrt{15} \lambda_1 + 14\sqrt{15} \lambda_2 + 20\sqrt{6} \lambda_4 + 13\sqrt{15} \lambda_5 \right)^{-1/2}. \quad (4.4)$$

In what follows, we will choose v to be real and positive without loss of generality. The physical T_7 states of the gauge bosons are complex linear combinations of the A_a^μ 's. The massive and T_7 charged gauge bosons are given by

$$Z^\mu = \frac{1}{\sqrt{2}} (A_7^\mu - i A_8^\mu), \quad (4.5a)$$

$$W_1^\mu = \frac{1}{\sqrt{2}} (A_4^\mu - i A_1^\mu), \quad (4.5b)$$

$$W_2^\mu = \frac{1}{\sqrt{2}} (A_5^\mu - i A_2^\mu), \quad (4.5c)$$

$$W_3^\mu = \frac{i}{\sqrt{2}} (A_6^\mu - i A_3^\mu), \quad (4.5d)$$

They obtain masses

$$m_Z^2 = \frac{7}{3} g^2 v^2 \quad \text{and} \quad m_W^2 = g^2 v^2. \quad (4.6)$$

The physical scalars arising from ϕ are mixtures of the fields listed in (4.2). For the one-dimensional representations one finds

$$\text{Re } \sigma_0 = \frac{1}{\sqrt{2}} (\phi_1 + \phi_1^*), \quad \text{Im } \sigma_0 = -\frac{i}{\sqrt{2}} (\phi_1 - \phi_1^*), \quad (4.7a)$$

$$\sigma_1 = \phi_2, \quad (4.7b)$$

with masses

$$m_{\text{Re } \sigma_0}^2 = 2\mu^2, \quad m_{\text{Im } \sigma_0}^2 = 0, \quad (4.8a)$$

$$m_{\sigma_1}^2 = -\mu^2 + \sqrt{15} \lambda_5 v^2. \quad (4.8b)$$

The massless mode can be understood noting that $\text{Im } \sigma_0$ is the Goldstone boson of an additional global $U(1)$ symmetry of the potential (2.2) which is spontaneously broken by $\langle \phi \rangle$. This symmetry prohibits a possible cubic coupling term for ϕ . The massless mode can be avoided by softly breaking the $U(1)$ via a reintroduction of the cubic term. Alternatively one could also gauge the additional $U(1)$ upon which the would-be Goldstone boson $\text{Im } \sigma_0$ gets eaten by the $U(1)$ gauge boson. Either way, this mode does not play any role in our discussion.

In contrast to the one dimensional representations, the triplet representations appear in identical copies. Therefore, the physical states are mixtures of T_1 , T_2 , and \bar{T}_3^* , with

mixing parameters depending on the potential parameters λ_i . The physical states are given by

$$\begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} = \underbrace{\begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix}}_{=V} \begin{pmatrix} T_2 \\ \overline{T}_3^* \\ T_1 \end{pmatrix}. \quad (4.9)$$

The mixing matrix V is an orthogonal matrix with entries depending on the potential parameters $V_{ij} = V_{ij}(\lambda_k)$ which are too complicated to be displayed here.⁴ Instead of discussing the most general case, we will settle to a specific set of potential parameters, which is sufficient to prove our point. We choose the parameter values⁵

$$\lambda_1 = 0.1, \quad \lambda_2 = -0.2, \quad \lambda_3 = 0, \quad \lambda_4 = 0, \quad \lambda_5 = 1, \quad (4.10)$$

which lead to a global T_7 minimum of the potential (2.2) with a VEV $v^2 \approx \mu^2 \times 0.856$. The corresponding mixing matrix of the physical states is given by

$$V \approx \begin{pmatrix} 0.950 & 0.288 & 0.121 \\ -0.304 & 0.941 & 0.146 \\ -0.072 & -0.175 & 0.982 \end{pmatrix}, \quad (4.11)$$

and the masses of the physical states are

$$m_{\tau_1}^2 \approx \mu^2 \times 0.946, \quad m_{\tau_2}^2 \approx \mu^2 \times 0.322, \quad m_{\tau_3}^2 \approx \mu^2 \times 0.142, \quad (4.12)$$

as well as

$$m_{\sigma_1}^2 \approx \mu^2 \times 2.316. \quad (4.13)$$

The hierarchies appearing here originate from the mild hierarchies in $\lambda_{1,2}/\lambda_5$.

5 Physical \mathcal{CP} violation in the T_7 phase

Given the physical fields, the stage is set for an explicit proof of physical \mathcal{CP} violation in the broken phase. The conserved outer automorphism, which corresponds to the \mathcal{CP} transformation at the level of $SU(3)$, acts on the physical states as

$$\text{Out}(T_7) : \begin{aligned} Z_\mu(x) &\mapsto -\mathcal{P}_\mu^\nu Z_\nu(\mathcal{P}x), & \sigma_0(x) &\mapsto \sigma_0(\mathcal{P}x), \\ W_\mu(x) &\mapsto \mathcal{P}_\mu^\nu W_\nu^*(\mathcal{P}x), & \sigma_1(x) &\mapsto \sigma_1(\mathcal{P}x), & \tau_i(x) &\mapsto \tau_i^*(\mathcal{P}x). \end{aligned} \quad (5.1)$$

Clearly, this does not correspond to a physical \mathcal{CP} transformation.

⁴While the mixing of physical states is dependent on the potential parameters, we note that the composition of the unphysical Goldstone bosons is independent of the potential parameters. This allows for the general choice (4.1).

⁵One should not be bothered by the fact that $\lambda_{3,4} = 0$, as there are also global T_7 minima for parameter choices $\lambda_i \neq 0 \forall i$.

This is easy to see for the quantum theory, where

$$\hat{\sigma}_1(x) = \int \widetilde{dp} \left\{ \hat{a}(\vec{p}) e^{-ipx} + \hat{b}^\dagger(\vec{p}) e^{ipx} \right\}, \quad (5.2)$$

and the transformation (5.1) corresponds to a map

$$\text{Out}(T_7) : \hat{a}(\vec{p}) \mapsto \hat{a}(-\vec{p}) \quad \text{and} \quad \hat{b}^\dagger(\vec{p}) \mapsto \hat{b}^\dagger(-\vec{p}). \quad (5.3)$$

In contrast, a physical \mathcal{CP} transformation of the complex scalar field $\sigma_1(x)$ would be a map $\sigma_1(x) \mapsto \sigma_1^*(x)$, corresponding to (see e.g. [19, 20])

$$\mathcal{CP} : \hat{a}(\vec{p}) \mapsto \hat{b}(-\vec{p}) \quad \text{and} \quad \hat{b}^\dagger(\vec{p}) \mapsto \hat{a}^\dagger(-\vec{p}). \quad (5.4)$$

It is straightforward to confirm that this naïve physical \mathcal{CP} transformation of the T_7 states, or any generalization thereof, is not a symmetry of the action. In fact, this directly follows from the fact that no class-inverting automorphism exists for T_7 . This already shows that physical \mathcal{CP} is violated in the T_7 phase.

In order to see this more explicitly, we construct \mathcal{CP} -odd basis invariants and show that they give rise to an observable \mathcal{CP} asymmetry

$$\varepsilon_{\sigma_1 \rightarrow W W^*} := \frac{|\mathcal{M}(\sigma_1 \rightarrow W W^*)|^2 - |\mathcal{M}(\sigma_1^* \rightarrow W W^*)|^2}{|\mathcal{M}(\sigma_1 \rightarrow W W^*)|^2 + |\mathcal{M}(\sigma_1^* \rightarrow W W^*)|^2}, \quad (5.5)$$

in the decay of a heavy charged scalar σ_1 to a pair of mutually conjugate heavy gauge boson triplets. Here $\mathcal{M}(i \rightarrow f)$ denotes the corresponding matrix element.

In a perturbative expansion, \mathcal{CP} violation arises from interference terms of diagrams that feature physical \mathcal{CP} -odd phases (cf. e.g. [19]). Physical observables must be independent of basis choices for all internal spaces and, therefore, can only depend on basis invariant quantities. An alternative way to a diagrammatic expansion, thus, is to construct basis invariants directly. The basis invariant approach is eminently useful and widely used in the context of \mathcal{CP} violation for example in the standard model [21, 22], but also in extensions with multiple families [23], additional scalars [24–29] or for theories with discrete symmetries [14, 30].⁶ The reason is that \mathcal{CP} -odd invariants can often be constructed without the need of performing involved calculations, even if \mathcal{CP} violation is arising only at higher loop order.⁷ In addition, it has been argued that the appearance of a single \mathcal{CP} odd basis invariant is enough in order to show that a model is \mathcal{CP} violating. To the best of our knowledge, however, it is not known how basis invariants are related to physical observables in general. This means that even if \mathcal{CP} odd invariants arise in a given model, it is still a logical possibility that the invariants delicately cancel against one another in all possible processes. Therefore, in addition to constructing specific \mathcal{CP} -odd basis invariants we also give a specific process for which we have checked that the invariants do not (all) cancel against one another.

⁶See [31] for comments regarding the conclusions of [14].

⁷We emphasize that this argument applies to outer automorphisms in general, not only for the case of \mathcal{CP} .

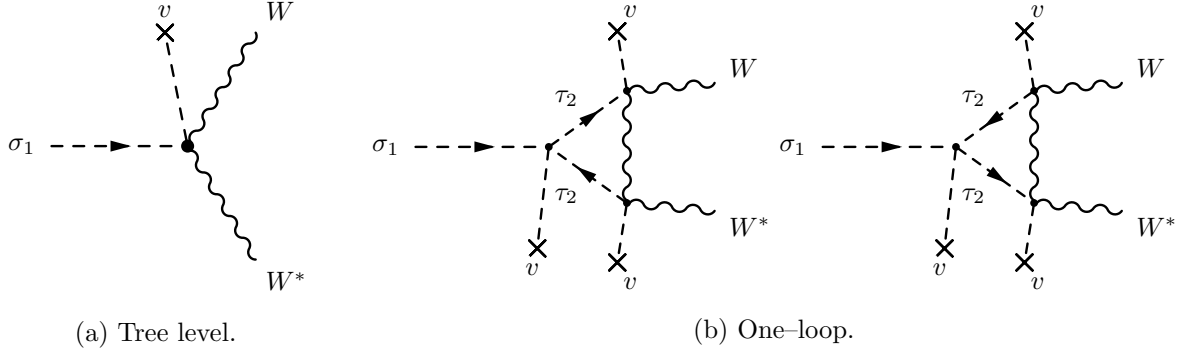


Figure 1. Diagrams whose interference gives rise to \mathcal{CP} violation in the decay of σ_1 to a pair of charged gauge bosons W .

For the choice of parameters given in equation (4.10) the decay $\sigma_1 \rightarrow W W^*$ is kinematically allowed if $g \lesssim 0.822$. The relevant couplings can be obtained by deriving the Lagrangean in the broken phase after the physical fields. The tree-level coupling of σ_1 to the charged gauge bosons is given by

$$[Y_{\sigma_1 W W^*}]_{ij} = \frac{\partial^3 \mathcal{L}}{\partial \sigma_1 \partial W_{\mu,i} \partial W_{\mu,j}^*} \propto v g^2. \quad (5.6)$$

Loop corrections to this vertex are possible, for example with triplets τ_2 in the loop (cf. Figure 1). The relevant couplings for this correction are given by

$$[Y_{\sigma_1 \tau_2 \tau_2^*}]_{ij} = \frac{\partial^3 \mathcal{L}}{\partial \sigma_1 \partial \tau_{2,i} \partial \tau_{2,j}^*} \propto v, \quad \text{and} \quad (5.7a)$$

$$[Y_{\tau_2^* W W^*}]_{ijk} = \frac{\partial^3 \mathcal{L}}{\partial \tau_{2,i}^* \partial W_{\mu,j} \partial W_{\mu,k}^*} \propto v g^2. \quad (5.7b)$$

While $Y_{\sigma_1 W W^*}$ can easily be stated in a closed form, $Y_{\sigma_1 \tau_2 \tau_2^*}$ and $Y_{\tau_2^* W W^*}$ are in general complicated functions of the potential parameters λ_i , see Equations (D.4) and (D.5) in Appendix D, where we also give general expression of the couplings independently of the potential parameters. From these, in general basis dependent, couplings it is straightforward to construct basis invariant quantities via contractions. For that, indices should be contracted such that basis transformations cancel.⁸ From the given couplings, we find two \mathcal{CP} -odd basis invariant contractions

$$\begin{aligned} \mathcal{I}_1 &= \left[Y_{\sigma_1 W W^*}^\dagger \right]_{k\ell} [Y_{\sigma_1 \tau_2 \tau_2^*}]_{ij} [Y_{\tau_2^* W W^*}]_{imk} [(Y_{\tau_2^* W W^*})^*]_{jml}, \quad \text{and} \\ \mathcal{I}_2 &= \left[Y_{\sigma_1 W W^*}^\dagger \right]_{k\ell} [Y_{\sigma_1 \tau_2 \tau_2^*}]_{ij} [Y_{\tau_2^* W W^*}]_{ilm} [(Y_{\tau_2^* W W^*})^*]_{jkm}. \end{aligned} \quad (5.8)$$

⁸The basis for each field can, in general, be rotated independently. However, assuming canonically normalized kinetic terms, basis transformations cancel in contractions of a field with its own complex conjugate.

The decay asymmetry $\varepsilon_{\sigma_1 \rightarrow WW^*}$ (cf. Equation (5.5)) receives contributions proportional to $2 \operatorname{Im} \mathcal{I}_{1,2} = (\mathcal{I}_{1,2} - \mathcal{I}_{1,2}^*)$. For our choice of parameters one finds

$$\operatorname{Im} \mathcal{I}_1 = +0.090 v^4 g^6 \quad \text{and} \quad \operatorname{Im} \mathcal{I}_2 = -0.126 v^4 g^6, \quad (5.9)$$

clearly indicating the presence of \mathcal{CP} violation. Inspecting the general expressions for the invariants (5.8) together with the general expressions of the couplings in Appendix D, we note that all contributing complex phases are parameter independent and arise from the projection of SU(3) Clebsch–Gordan coefficients onto the T_7 subgroup. However, the geometric phases are weighted by functions of the continuous parameters λ_i of the potential. Therefore, the resulting phases of \mathcal{I}_1 and \mathcal{I}_2 depend on the potential parameters. Furthermore, we note that the two invariants are closely related to each other by the relation

$$\mathcal{I}_1 = \omega \mathcal{I}_2, \quad (5.10)$$

which is dictated by the $T_7 \rtimes \mathbb{Z}_2$ symmetry.

There are additional contributions to $\varepsilon_{\sigma_1 \rightarrow WW^*}$ from other one loop diagrams, for example those containing $\tau_{1,3}$ or gauge bosons running in the loop. We have explicitly checked that these contributions do not cancel the asymmetry.

Note that all relevant couplings as well as the decay asymmetry are proportional to positive powers of v . Therefore, there is no physical \mathcal{CP} violation when the SU(3) symmetry is restored by taking the limit $v \rightarrow 0$.

6 Natural protection of $\theta = 0$ in the broken phase

Finally, let us observe how a possible θ -term [32–34] is affected if \mathcal{CP} violation arises in the way described above. The usual topological term

$$\mathcal{L}_\theta = \theta \frac{g^2}{32\pi^2} G_{\mu\nu}^a \tilde{G}^{\mu\nu,a}, \quad (6.1)$$

where $G_{\mu\nu}^a$ is the field strength and $\tilde{G}^{\mu\nu,a} := \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} G_{\rho\sigma}^a$ its dual, is odd under parity or time-reversal transformations. Therefore, $\theta = 0$ is enforced by the transformation (2.3a) and \mathcal{L}_θ is absent from the theory. Crucially, the transformation (2.3a) is not broken by the VEV. Thus, even though it is not possible to interpret the transformation (2.3a) as a physical \mathcal{CP} transformation of the physical states in the broken phase, the transformation is unbroken and warrants that $\theta = 0$, not only for the broken SU(3), but also for other gauge groups. We stress that our model is not realistic and does not even contain fermions. Nevertheless, it is tempting to speculate that the observed mechanism does provide a symmetry based solution to the strong \mathcal{CP} problem in more realistic settings.

For example, one may construct models based on $[G_{\text{SM}} \times \text{SU}(3)_{\text{F}}] \rtimes \mathbb{Z}_2^{\mathcal{CP}}$, where the flavor symmetry $\text{SU}(3)_{\text{F}}$ gets broken spontaneously to T_7 , or another type I group, without breaking $\mathbb{Z}_2^{\mathcal{CP}}$. In an intermediate step, one would have $[G_{\text{SM}} \times T_7] \rtimes \mathbb{Z}_2$, where the \mathbb{Z}_2 symmetry continues to forbid θ_{QCD} , while physical \mathcal{CP} is violated in the flavor sector. Of course, the \mathbb{Z}_2 must then also be preserved when the remaining T_7 symmetry is spontaneously broken. A detailed study of this new avenue in flavor model building is beyond the scope of this work, but will be explored elsewhere.

7 Summary and Discussion

We have studied how one obtains a T_7 toy model from a \mathcal{CP} conserving $SU(3)$ theory by spontaneous breaking. T_7 is a so-called type I group, i.e. it is not possible to impose a physical \mathcal{CP} transformation on a T_7 model with generic field content while maintaining the interactions of the theory. This reflects the fact that there is no basis for T_7 in which all Clebsch–Gordan coefficients are real.

We paid particular attention to the fate of the $SU(3)$ – \mathcal{CP} transformation. We found that it is *not* spontaneously broken by the **15**–plet VEV which breaks $SU(3)$ to T_7 . Rather, the $SU(3)$ – \mathcal{CP} transformation corresponds to the unique \mathbb{Z}_2 outer automorphism of T_7 in the broken phase. This automorphism does *not* warrant physical \mathcal{CP} conservation for the T_7 states, and there is also no other possible outer automorphism which could do the job. Thus, \mathcal{CP} is violated in the broken phase.

Stated in simple terms, \mathcal{CP} is violated because there are T_7 states which emerge as *complex* linear combinations of certain $SU(3)$ states. An example is the Z boson, which transforms as complex T_7 **1₁**–plet but does *not* get conjugated under the T_7 outer automorphism transformation (cf. Figure 2). That is, the outer automorphism cannot be interpreted as a \mathcal{CP} transformation at the level of T_7 , and physical \mathcal{CP} is violated. We have demonstrated this explicitly by establishing a decay asymmetry in the decay of a complex scalar to massive gauge bosons.

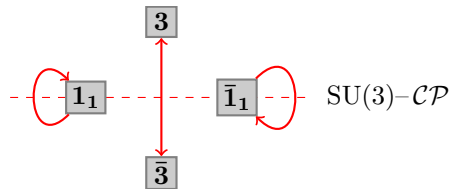


Figure 2. Action of $SU(3)$ – \mathcal{CP} on the T_7 multiplets contained in the adjoint of $SU(3)$. A transformation that exchanges both $\mathbf{1}_1 \leftrightarrow \bar{\mathbf{1}}_1$ and $\mathbf{3} \leftrightarrow \bar{\mathbf{3}}$ is inconsistent with T_7 and $SU(3)$.

Our findings have interesting physical consequences. The definition of matter and antimatter, at least with respect to a \mathcal{CP} mirror, is not universally possible for chains of groups and subgroups. Rather, the definition of matter and antimatter depends on the underlying unbroken symmetry. This has profound implications for cosmology, where the symmetries of the ground state change in the course of the evolution of the universe.

Interestingly, the θ parameters of $SU(3)$ and other gauge groups remain forbidden by the outer automorphism, also in the broken phase. This may allow one to construct realistic models, in which \mathcal{CP} is broken in the flavor sector, but $\bar{\theta}_{\text{QCD}}$ is forbidden by the outer automorphism.

In our analysis, we have restricted ourselves to only one simple Lie group, $SU(3)$, and one type I symmetry, T_7 . It will be interesting to generalize the discussion to other groups with richer outer automorphism structure and include fermions in the discussion.

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A Details of the quartic SU(3) invariants

There are in total 14 linearly independent SU(3) invariants in the contraction $\overline{\mathbf{15}} \otimes \mathbf{15} \otimes \overline{\mathbf{15}} \otimes \mathbf{15}$. We use the `SusyNo Mathematica` package [17] to compute them via the command

$$\text{Invariants}[\text{SU3}, \{\{2, 1\}, \{1, 2\}, \{2, 1\}, \{1, 2\}\}] . \quad (\text{A.1})$$

`SusyNo` provides the invariants ordered according to their permutation group representations and we adopt that ordering. Only the first five invariants are non-vanishing if all $\mathbf{15}$ -plets correspond to the same field ϕ . In this ordering, the invariants are multiplied by factors λ_i , $i = 1, \dots, 5$, resulting in the quartic part of the potential (2.2).

B Details of SU(3) and the T_7 -diagonal basis

The basis we choose for the SU(3) triplet generators has been given by Fonseca [35] and it is implemented in the `SusyNo Mathematica` package [17]. The generators are given by

$$\begin{aligned} \mathbf{t}_1^{(3)} &= \begin{pmatrix} & \frac{1}{2} \\ \frac{1}{2} & \end{pmatrix}, & \mathbf{t}_2^{(3)} &= \begin{pmatrix} & \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, & \mathbf{t}_3^{(3)} &= \begin{pmatrix} & \frac{i}{2} \\ -\frac{i}{2} & \end{pmatrix}, \\ \mathbf{t}_4^{(3)} &= \begin{pmatrix} & -\frac{i}{2} \\ \frac{i}{2} & \end{pmatrix}, & \mathbf{t}_5^{(3)} &= \begin{pmatrix} & \\ \frac{i}{2} & -\frac{i}{2} \end{pmatrix}, & \mathbf{t}_6^{(3)} &= \begin{pmatrix} & -\frac{1}{2} \\ -\frac{1}{2} & \end{pmatrix}, \end{aligned} \quad (\text{B.1})$$

$$\mathbf{t}_7^{(3)} = \frac{1}{2\sqrt{3}} \begin{pmatrix} 2 & & \\ & -1 & \\ & & -1 \end{pmatrix}, \quad \mathbf{t}_8^{(3)} = \begin{pmatrix} & \\ \frac{1}{2} & \\ & -\frac{1}{2} \end{pmatrix}.$$

In order to obtain a basis in which the T_7 elements A and B are block-diagonal we rotate the (ij) -space) basis of the generators given in `SusyNo` according to

$$\mathbf{t}_a^{(15, T_7)} = V_{15}^\dagger \mathbf{t}_a^{(15)} V_{15}. \quad (\text{B.2})$$

The corresponding rotation matrix is given by

$$V_{15} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & 1 & 0 \\ -1 & e^{-i\pi/3} & e^{i\pi/3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \sqrt{2} \\ 1 & e^{-2i\pi/3} & e^{i\pi/3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\sqrt{2/3} & 0 & 0 & -1/\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/\sqrt{3} & 0 & 0 & -2\sqrt{2/3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -\sqrt{2} & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{B.3})$$

Due to the degeneracy of T_7 representations in the **15**-plet, we note that there is a degeneracy in this basis choice corresponding to rotations among T_7 triplets and among anti-triplets, respectively. We have chosen our basis such that the (anti-)triplet Goldstone modes reside only in one of the (anti-)triplets, thereby allowing for the simplest form of Equation (4.1).

In order to obtain a basis for the adjoint space in which the T_7 elements A and B are block-diagonal we rotate the (a -space) basis of the generators according to⁹

$$\mathbf{t}_a^{(8, T_7)} = V_{\mathbf{8}, ab}^T \mathbf{t}_b^{(8)}. \quad (\text{B.4})$$

The corresponding transformation matrix is given by

$$V_{\mathbf{8}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & i & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 & 0 & i \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ i & -i & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{B.5})$$

This, of course, is consistent with (4.5), where

$$(Z^\mu, Z^{\mu*}, W_1^\mu, W_2^\mu, W_3^\mu, W_1^{\mu*}, W_2^{\mu*}, W_3^{\mu*})_a^T = [V_{\mathbf{8}}^\dagger]_{ab} A_b^\mu. \quad (\text{B.6})$$

C Details of T_7

The group T_7 has been used in flavor model building [36–44] also motivated by the fact that T_7 is the smallest finite subgroup of $SU(3)$ which has an irreducible triplet representation. T_7 is the smallest non-Abelian finite subgroup of $SU(3)$ which is not also a subgroup of $SU(2)$ or $SO(3)$ (cf. e.g. [16]). Further details of T_7 can, for example, be found in [45], whose conventions we follow. T_7 is implemented in **GAP** [46] as SmallGroup SG(21, 1).

The outer automorphism group of T_7 has order two, and is generated by the transformation

$$u : (a, b) \mapsto (a^6, b). \quad (\text{C.1})$$

The action of the outer automorphism on the irreps has already been given in (2.6), and it is clearly not class-inverting. We note that the chosen basis (3.2) is an eigenbasis of the outer automorphism, meaning that the consistency condition [9, 47, 48]

$$U A^* U^{-1} = A^6, \quad U B^* U^{-1} = B, \quad (\text{C.2})$$

is solved by $U = \mathbb{1}$.

⁹The transformation of the gauge bosons is always given by $[\mathbf{t}_a^{(8)}]_{ij} = f_{aij}$.

D Couplings

Here we give general expressions for the couplings defined in equations (5.6), (5.7a), and (5.7b). The couplings turn out to be

$$Y_{\sigma_1 WW^*} = \frac{v g^2}{\sqrt{6}} e^{-\pi i/6} \text{diag}(1, \omega, \omega^2), \quad (\text{D.1a})$$

$$Y_{\sigma_1 \tau_2 \tau_2^*} = v y_{\sigma_1 \tau_2 \tau_2^*} \text{diag}(1, \omega, \omega^2), \quad (\text{D.1b})$$

with the usual $\omega = e^{2\pi i/3}$, and

$$[Y_{\tau_2^* WW^*}]_{121} = [Y_{\tau_2^* WW^*}]_{232} = [Y_{\tau_2^* WW^*}]_{313} = v g^2 y_{\tau_2^* WW^*}, \quad (\text{D.2})$$

with

$$[Y_{\tau_2^* WW^*}]_{ijk} = 0, \quad (\text{D.3})$$

for all other choices of indices. For a general choice of parameters, the values of the couplings are given by

$$\begin{aligned} y_{\sigma_1 \tau_2 \tau_2^*} = \frac{1}{504 \sqrt{3}} \Big\{ & V_{21}^2 \left[-14 \sqrt{10} (17 + 5 \sqrt{3} i) \lambda_1 + 84 \sqrt{30} (\sqrt{3} - i) \lambda_2 \right. \\ & \left. - 240 (1 + \sqrt{3} i) \lambda_4 - \sqrt{10} (197 - 55 \sqrt{3} i) \lambda_5 \right] \\ & + 8 V_{22}^2 \left[28 \sqrt{10} (1 - \sqrt{3} i) \lambda_1 - 14 \sqrt{30} i \lambda_2 + 112 \sqrt{3} i \lambda_3 \right. \\ & \left. - (30 - 26 \sqrt{3} i) \lambda_4 + \sqrt{10} (20 - \sqrt{3} i) \lambda_5 \right] \\ & + 8 V_{23}^2 \left[28 \sqrt{10} (1 + \sqrt{3} i) \lambda_1 - 14 \sqrt{30} i \lambda_2 - 168 \lambda_3 \right. \\ & \left. + (6 + 65 \sqrt{3} i) \lambda_4 - 4 \sqrt{10} (1 - 2 \sqrt{3} i) \lambda_5 \right] \\ & + 8 V_{21} V_{22} \left[-35 \sqrt{10} (1 - \sqrt{3} i) \lambda_1 + 21 \sqrt{30} (\sqrt{3} + i) \lambda_2 \right. \\ & \left. - 56 (3 + \sqrt{3} i) \lambda_3 + 6 (1 + 17 \sqrt{3} i) \lambda_4 - \sqrt{10} (67 + 19 \sqrt{3} i) \lambda_5 \right] \\ & + 4 V_{21} V_{23} \left[-28 \sqrt{10} (2 + \sqrt{3} i) \lambda_1 - 42 \sqrt{30} (\sqrt{3} + i) \lambda_2 \right. \\ & \left. + 30 (11 + 3 \sqrt{3} i) \lambda_4 - \sqrt{10} (31 + 11 \sqrt{3} i) \lambda_5 \right] \\ & \left. - 8 V_{22} V_{23} \left[14 \sqrt{10} \lambda_1 - 14 \sqrt{30} i \lambda_2 \right. \right. \\ & \left. \left. + 10 (3 + 5 \sqrt{3} i) \lambda_4 + \sqrt{10} (1 - 3 \sqrt{3} i) \lambda_5 \right] \right\} \end{aligned} \quad (\text{D.4})$$

and

$$y_{\tau_2^* WW^*} = -\frac{\sqrt{2}}{3} (2 V_{21} + V_{22} + 2 V_{23}). \quad (\text{D.5})$$

For the choice of parameters given in (4.10), one finds numerical values

$$y_{\sigma_1 \tau_2 \tau_2^*} \approx 1.181 + 0.298 i \quad \text{and} \quad y_{\tau_2^* WW^*} \approx -0.295. \quad (\text{D.6})$$

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